Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 18 December 2017, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of each of your exam sheets and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you **must** explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 4 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (20 points)

Consider the function

$$f(z) = \frac{\bar{z}}{1-z}.$$

(a) (8 points) Write $f(z)$ in the form $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$.

Solution

We compute

$$\begin{aligned} f(z) &= \frac{\bar{z}}{1-z} = \frac{x-iy}{1-x-iy} = \frac{(x-iy)(1-x+iy)}{(1-x-iy)(1-x+iy)} \\ &= \frac{x-x^2+y^2}{(1-x)^2+y^2} + i\frac{-y+2xy}{(1-x)^2+y^2}. \end{aligned}$$

We identify

$$u = \frac{x-x^2+y^2}{(1-x)^2+y^2}, \quad v = \frac{-y+2xy}{(1-x)^2+y^2}.$$

(b) (12 points) Use the Cauchy-Riemann equations to determine where $f(z)$ is differentiable.

Solution

We check the Cauchy-Riemann equations. We have

$$\frac{\partial u}{\partial x} = \frac{1-2x+x^2+3y^2-4xy^2}{((1-x)^2+y^2)^2},$$

and

$$\frac{\partial v}{\partial y} = \frac{-1+4x-5x^2+2x^3+y^2-2xy^2}{((1-x)^2+y^2)^2}.$$

For the first Cauchy-Riemann equation to hold we need

$$\begin{aligned}
0 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\
&= \frac{2 - 6x + 6x^2 - 2x^3 + 2y^2 - 2xy^2}{((1-x)^2 + y^2)^2} \\
&= 2\frac{1 - 3x + 3x^2 - x^3 + (1-x)y^2}{((1-x)^2 + y^2)^2} \\
&= 2\frac{(1-x)^3 + (1-x)y^2}{((1-x)^2 + y^2)^2} \\
&= 2(1-x)\frac{(1-x)^2 + y^2}{((1-x)^2 + y^2)^2} \\
&= \frac{2(1-x)}{(1-x)^2 + y^2},
\end{aligned}$$

which gives $x = 1$.

Moreover, we compute

$$\frac{\partial u}{\partial y} = \frac{2y - 6xy + 4x^2y}{((1-x)^2 + y^2)^2},$$

and

$$\frac{\partial v}{\partial x} = \frac{2xy - 2x^2y + 2y^3}{((1-x)^2 + y^2)^2}.$$

For the second Cauchy-Riemann equation to hold we need

$$\begin{aligned}
0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
&= \frac{2y - 4xy + 2x^2y + 2y^3}{((1-x)^2 + y^2)^2} \\
&= 2y\frac{1 - 2x + x^2 + y^2}{((1-x)^2 + y^2)^2} \\
&= \frac{2y}{(1-x)^2 + y^2},
\end{aligned}$$

which gives $y = 0$.

Therefore, we see that the Cauchy-Riemann equations are simultaneously satisfied only at $z = x + iy = 1$. However, the function $f(z)$ is not defined at $z = 1$. This means that $f(z)$ is nowhere differentiable.

Question 2 (20 points)

The principal value of arcsin is defined as

$$\operatorname{Arcsin}(z) = -i \operatorname{Log}\left(iz + \sqrt{1 - z^2}\right),$$

where $\sqrt{z}$ denotes the principal value of $z^{1/2}$ (consider known that: for $x > 0$, $\sqrt{x}$ equals the real square root; for $x < 0$, $\sqrt{x} = i\sqrt{|x|}$; and that $\sqrt{0} = 0$).

(a) (10 points) Compute $\operatorname{Arcsin}(1)$, $\operatorname{Arcsin}(i)$, and $\operatorname{Arcsin}(2)$.

Solution

We first compute

$$\begin{aligned}\text{Arcsin}(1) &= -i\,\text{Log}\left(i + \sqrt{1-1^2}\right)\\ &= -i\,\text{Log}(i)\\ &= -i\big(\,\text{Log}\,|i| + i\,\text{Arg}(i)\big)\\ &= -i\Big(\,\text{Log}\,1 + i\frac{\pi}{2}\Big)\\ &= \frac{\pi}{2}.\end{aligned}$$

Then we compute

$$\begin{aligned}\text{Arcsin}(i) &= -i\,\text{Log}\left(i^2 + \sqrt{1-i^2}\right)\\ &= -i\,\text{Log}\left(-1+\sqrt{2}\right),\end{aligned}$$

and we not need any other computations since for $x > 0$, the principal value $\text{Log}\, x$ of the complex logarithm is exactly the real logarithm of x.

Finally,

$$\begin{aligned}\text{Arcsin}(2) &= -i\,\text{Log}\left(2i + \sqrt{1-2^2}\right)\\ &= -i\,\text{Log}\left(2i+\sqrt{3}i\right)\\ &= -i\left(\text{Log}\,|2i+\sqrt{3}i| + i\,\text{Arg}(2i+\sqrt{3}i)\right)\\ &= -i\left(\text{Log}\,|2+\sqrt{3}| + i\frac{\pi}{2}\right)\\ &= \frac{\pi}{2} - i\,\text{Log}\,|2+\sqrt{3}|.\end{aligned}$$

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(b) (10 points) Show that the half-line on the complex plane defined by $z \in \mathbb{R}$ with $z > 1$ is a branch cut of Arcsin.

Solution

Alternative 1. The principal value, $\sqrt{z}$, of $z^{1/2}$ has a branch cut along the negative real axis and the limits from different sides of the cut are $\pm i\sqrt{|z|}$. For $z > 1$ we have $1 - z^2 < 0$. Therefore, the function $\sqrt{1-z^2}$ has a branch cut for $z > 1$ and the two limits are $\pm i\sqrt{z^2-1}$. We conclude that $\text{Arcsin}(z)$ also has a branch cut for $z > 1$ and the two limits are $-i\,\text{Log}(iz \pm i\sqrt{z^2-1})$ (from continuity of Log along the imaginary axis). Note that the limits are distinct since the two numbers $z \pm \sqrt{z^2-1}$ are distinct (for $z > 1$) and Log is one-to-one.

Alternative 2. We consider the limits $\lim_{h\to 0^\pm} \text{Arcsin}(z+ih)$ for $z > 1$. We first note that

$$\begin{aligned}\lim_{h\to 0^\pm} \text{Log}(1-(z+ih)^2) &= \lim_{h\to 0^\pm} \text{Log}(1-z^2+h^2-2izh)\\ &= \lim_{h\to 0^\pm} \text{Log}\,|1-z^2+h^2-2izh| + i\lim_{h\to 0^\pm} \text{Arg}(1-z^2+h^2-2izh)\\ &= \text{Log}\,|1-z^2| + i\lim_{h\to 0^\pm} \text{Arg}(1-z^2+h^2-2izh)\end{aligned}$$

(where we used that $|z|$ is continuous on $\mathbb{C}$ and Log is continuous for (real) positive numbers)

$$= \operatorname{Log}|1 - z^2| \mp \pi i$$

(where we used that as $h \to 0^+$ the number $1 - z^2 + h^2 - 2izh$ approaches the (real) negative number $1 - z^2$ from below the negative semi-axis, while for $h \to 0^-$ from above the negative semi-axis).

The function $\exp(z)$ is continuous on $\mathbb{C}$, thus

$$\begin{aligned}
\lim_{h\to 0^\pm} \sqrt{1-(z+ih)^2} &= \lim_{h\to 0^\pm} \exp\left(\frac{1}{2}\operatorname{Log}(1-(z+ih)^2)\right) \\
&= \exp\left(\frac{1}{2}\lim_{h\to 0^\pm}\operatorname{Log}(1-(z+ih)^2)\right) \\
&= \exp\left(\frac{1}{2}\operatorname{Log}|1-z^2| \mp i\frac{\pi}{2}\right) \\
&= \exp\left(\mp i\frac{\pi}{2}\right)\exp\left(\frac{1}{2}\operatorname{Log}|1-z^2|\right) \\
&= \mp i\exp\left(\frac{1}{2}\operatorname{Log}|1-z^2|\right) \\
&= \mp i\sqrt{|1-z^2|} \\
&= \mp i\sqrt{z^2-1}.
\end{aligned}$$

Finally, since Log is continuous along the imaginary axis we have

$$\begin{aligned}
\lim_{h\to 0^\pm}\operatorname{Arcsin}(z+ih) &= -i\lim_{h\to 0^\pm}\operatorname{Log}(i(z+ih)+\sqrt{1-(z+ih)^2}) \\
&= -i\operatorname{Log}(i\lim_{h\to 0^\pm}(z+ih)+\lim_{h\to 0^\pm}\sqrt{1-(z+ih)^2}) \\
&= -i\operatorname{Log}(iz \mp i\sqrt{z^2-1}) \\
&= -i\left(\operatorname{Log}|z\mp\sqrt{z^2-1}| + i\operatorname{Arg}(i(z\mp\sqrt{z^2-1}))\right) \\
&= \frac{\pi}{2} - i\operatorname{Log}|z\mp\sqrt{z^2-1}|.
\end{aligned}$$

Since the two limits are different for any z on the half-line $z > 1$ we conclude that the half-line is a branch cut for Arcsin.

Question 3 (20 points)

Consider the closed unit disk $U = \{z \in \mathbb{C} : |z| \leq 1\}$. Show that

$$\max_{z\in U}|az^n + b| = |a| + |b|.$$

Here $a, b \in \mathbb{C}$ are constant and n is an integer with $n \geq 1$.

Solution

The function $f(z) = az^n + b$ is analytic on $D = \{|z| < 1\}$ and continuous on U (actually, $f(z)$ is analytic and thus continuous on $\mathbb{C}$). Note that U is the union of D and the its boundary.

From the **maximum modulus principle** we conclude that $|f(z)|$ should attain its maximum value at the boundary $C = \{|z| = 1\}$ of D. It is therefore enough to look for such maximum value on C.

Alternative 1. For $z \in C$ we have

$$\begin{aligned} |f(z)|^2 &= |az^n + b|^2 \\ &= |az^n + b|^2 \\ &= (az^n + b)(\bar{a}\bar{z}^n + \bar{b}) \\ &= a\bar{a}z^n\bar{z}^n + b\bar{b} + a\bar{b}z^n + \bar{a}b\bar{z}^n \\ &= |a|^2|z|^{2n} + |b|^2 + 2\,\mathrm{Re}(a\bar{b}z^n) \\ &= |a|^2 + |b|^2 + 2\,\mathrm{Re}(a\bar{b}z^n). \end{aligned}$$

Since $z \in C$ write $z = e^{it}$ with $t \in \mathbb{R}$. Moreover, write $a = |a|e^{iu}$, $b = |b|e^{iv}$. Then

$$\begin{aligned} |f(e^{it})|^2 &= |a|^2 + |b|^2 + 2\,\mathrm{Re}(|a||b|\exp(i(u - v + nt))) \\ &= |a|^2 + |b|^2 + 2|a||b|\cos(u - v + nt). \end{aligned}$$

Therefore, for $z = e^{it} \in \mathbb{C}$, $|f(e^{it})|^2$ attains the maximum value $|a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2$ for some $z_* \in C$ (for example, choose $z_* = e^{it_*}$ with $t_* = (v - u)/n$). Since the (real) function x^2 (for $x \geq 0$) is strictly increasing we conclude that $|f(e^{it})|$ also attains its maximum value on C at the same z_* and it equals

$$|f(z_*)| = |a| + |b|.$$

Alternative 2. For $z \in C$ we have

$$|f(z)| \leq |az^n + b| \leq |az^n| + |b| = |a||z|^n + |b| = |a| + |b|.$$

This means that if we find a $z_* \in C$ such that $|f(z_*)| = |a| + |b|$ then that will be the maximum value of $|f(z)|$ on C. Writing as above $a = |a|e^{iu}$, $b = |b|e^{iv}$, we have

$$\begin{aligned} |f(z)| &= |az^n + b| \\ &= ||a|e^{iu}z^n + |b|e^{iv}| \\ &= |e^{iv}|||a|e^{i(u-v)}z^n + |b|| \\ &= ||a|e^{i(u-v)}z^n + |b||. \end{aligned}$$

Let

$$z_* = e^{i(v-u)/n} \in C \Rightarrow z_*^n = e^{i(v-u)}.$$

Then

$$\begin{aligned} |f(z_*)| &= ||a|e^{i(u-v)}z_*^n + |b|| \\ &= ||a|e^{i(u-v)}e^{i(v-u)} + |b|| \\ &= ||a| + |b|| \\ &= |a| + |b|, \end{aligned}$$

which is exactly what we wanted to show.

Question 4 (30 points)

(a) (15 points) Compute the value of the integral

$$\int_\Gamma \frac{e^z}{(z+1)(z^2-4)}\,dz,$$

where Γ is the closed contour shown in Figure 1.

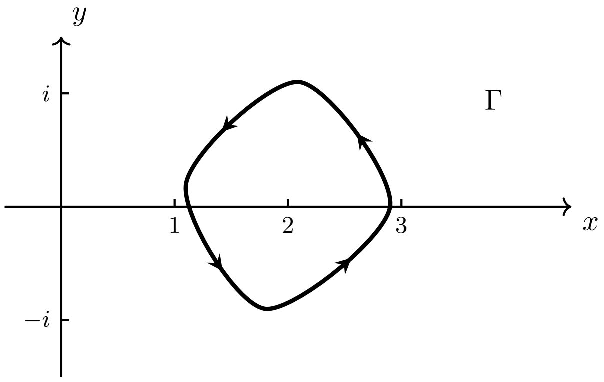

Figure 1: Contour Γ for Question 4(a).

Solution

The Cauchy integral formula for $z_0 = 2$ gives that for a function $g(z)$ analytic on and inside Γ we have

$$\int_\Gamma \frac{g(z)}{z-2}\,dz = 2\pi i g(2).$$

Choosing

$$g(z) = \frac{e^z}{(z+1)(z+2)},$$

we note that it is analytic on and inside Γ so it satisfies the conditions for applying the formula.

Therefore

$$\int_\Gamma \frac{e^z}{(z+1)(z^2-4)}\,dz = \int_\Gamma \frac{e^z}{(z+1)(z+2)(z-2)}\,dz = \int_\Gamma \frac{g(z)}{z-2}\,dz = 2\pi i g(2).$$

We then compute

$$g(2) = \frac{e^2}{(2+1)(2+2)} = \frac{e^2}{12}.$$

Finally,

$$\int_\Gamma \frac{e^z}{(z+1)(z^2-4)}\,dz = 2\pi i g(2) = \frac{e^2\pi i}{6}.$$

(b) (15 points) Compute the value of the integral

$$\int_C (\bar{z} + z^2 \sin z)\,dz,$$

where C is the circle $|z-1|=1$ traversed in the clockwise direction.

Solution

We have

$$\int_C (\bar{z}+z^2\sin z)\,dz = \int_C \bar{z}\,dz + \int_C z^2\sin z\,dz.$$

The function $z^2\sin z$ is entire and therefore its integral along any loop vanishes. Thus we only need to compute $\int_C \bar{z}\,dz$.

We parameterize C by $z(t)=1+e^{-it}$, $0\le t\le 2\pi$. Then $z'(t)=-ie^{-it}$ and $\overline{z(t)}=1+e^{it}$. We have

$$\begin{aligned}\int_C \bar{z}\,dz &= \int_0^{2\pi} \overline{z(t)}z'(t)\,dt\\ &= -i\int_0^{2\pi}(1+e^{it})e^{-it}\,dt\\ &= -i\int_0^{2\pi} e^{-it}\,dt - i\int_0^{2\pi} dt\\ &= \int_0^{2\pi}(e^{-it})'\,dt - 2\pi i\\ &= (e^{-2\pi i}-1)-2\pi i\\ &= (1-1)-2\pi i\\ &= -2\pi i.\end{aligned}$$

Finally,

$$\int_C (z^2\sin z+\bar{z})\,dz = 0-2\pi i = -2\pi i.$$